Problem 4.22

(a) Starting with the canonical commutation relations for position and momentum (Equation 4.10), work out the following commutators:

$$\begin{aligned} [L_z, x] &= i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0 \\ [L_z, p_x] &= i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0. \end{aligned} \tag{4.122}$$

(b) Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from Equation 4.96.

(c) Find the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where, of course, $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

(d) Show that the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r. (Thus H, L^2, and L_z are mutually compatible observables.)

Solution

Part (a)

Equation 4.10 on page 132 was derived in Problem 4.1.

$$[x_j, p_k] = i\hbar\delta_{jk} \quad \Rightarrow \quad \begin{cases} [x, p_x] = i\hbar, & [x, p_y] = 0, & [x, p_z] = 0 \\ [y, p_x] = 0, & [y, p_y] = i\hbar, & [y, p_z] = 0 \\ [z, p_x] = 0, & [z, p_y] = 0, & [z, p_z] = i\hbar \end{cases}$$

$$[x_j, x_k] = 0 \quad \Rightarrow \quad \begin{cases} [x, x] = 0, & [x, y] = 0, & [x, z] = 0 \\ [y, x] = 0, & [y, y] = 0, & [y, z] = 0 \\ [z, x] = 0, & [z, y] = 0, & [z, z] = 0 \end{cases}$$

$$[p_j, p_k] = 0 \quad \Rightarrow \quad \begin{cases} [p_x, p_x] = 0, & [p_x, p_y] = 0, & [p_x, p_z] = 0 \\ [p_y, p_x] = 0, & [p_y, p_y] = 0, & [p_y, p_z] = 0 \\ [p_z, p_x] = 0, & [p_z, p_y] = 0, & [p_z, p_z] = 0 \end{cases}$$

Recall the commutator identities from Problem 3.14 on page 108.

$$\left[\hat{A} + \hat{B}, \hat{C}\right] = \left[\hat{A}, \hat{C}\right] + \left[\hat{B}, \hat{C}\right]$$

$$\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$$

And note how the three components of $\mathbf{L}$ are defined.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \quad \Rightarrow \quad \begin{cases} L_x = yp_z - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{cases}$$

Derive the first commutator.

$$\begin{aligned}
[L_z, x] &= [xp_y - yp_x, x] \\
&= [xp_y, x] - [yp_x, x] \\
&= (x[p_y, x] + [x, x]p_y) - (y[p_x, x] + [y, x]p_x) \\
&= (-x[x, p_y] + 0 \cdot p_y) - (-y[x, p_x] + 0 \cdot p_x) \\
&= (-x \cdot 0) - (-y \cdot i\hbar) \\
&= i\hbar y
\end{aligned}$$

Derive the second commutator.

$$\begin{aligned}
[L_z, y] &= [xp_y - yp_x, y] \\
&= [xp_y, y] - [yp_x, y] \\
&= (x[p_y, y] + [x, y]p_y) - (y[p_x, y] + [y, y]p_x) \\
&= (-x[y, p_y] + 0 \cdot p_y) - (-y[y, p_x] + 0 \cdot p_x) \\
&= (-x \cdot i\hbar) - (-y \cdot 0) \\
&= -i\hbar x
\end{aligned}$$

Derive the third commutator.

$$\begin{aligned}
[L_z, z] &= [xp_y - yp_x, z] \\
&= [xp_y, z] - [yp_x, z] \\
&= (x[p_y, z] + [x, z]p_y) - (y[p_x, z] + [y, z]p_x) \\
&= (-x[z, p_y] + 0 \cdot p_y) - (-y[z, p_x] + 0 \cdot p_x) \\
&= (-x \cdot 0) - (-y \cdot 0) \\
&= 0
\end{aligned}$$

Derive the fourth commutator.

$$\begin{aligned}
[L_z, p_x] &= [xp_y - yp_x, p_x] \\
&= [xp_y, p_x] - [yp_x, p_x] \\
&= (x[p_y, p_x] + [x, p_x]p_y) - (y[p_x, p_x] + [y, p_x]p_x) \\
&= (x \cdot 0 + i\hbar \cdot p_y) - (y \cdot 0 + 0 \cdot p_x) \\
&= i\hbar p_y
\end{aligned}$$

Derive the fifth commutator.

$$\begin{aligned}
[L_z, p_y] &= [xp_y - yp_x, p_y] \\
&= [xp_y, p_y] - [yp_x, p_y] \\
&= (x[p_y, p_y] + [x, p_y]p_y) - (y[p_x, p_y] + [y, p_y]p_x) \\
&= (x \cdot 0 + 0 \cdot p_y) - (y \cdot 0 + i\hbar \cdot p_x) \\
&= -i\hbar p_x
\end{aligned}$$

Derive the sixth commutator.

$$\begin{aligned}
[L_z, p_z] &= [xp_y - yp_x, p_z] \\
&= [xp_y, p_z] - [yp_x, p_z] \\
&= (x[p_y, p_z] + [x, p_z]p_y) - (y[p_x, p_z] + [y, p_z]p_x) \\
&= (x \cdot 0 + 0 \cdot p_y) - (y \cdot 0 + 0 \cdot p_x) \\
&= 0
\end{aligned}$$

Part (b)

Use the results of part (a) to derive the commutator of L_z and L_x.

$$\begin{aligned}
[L_z, L_x] &= -[L_x, L_z] \\
&= -[yp_z - zp_y, L_z] \\
&= -([yp_z, L_z] - [zp_y, L_z]) \\
&= -\{(y[p_z, L_z] + [y, L_z]p_z) - (z[p_y, L_z] + [z, L_z]p_y)\} \\
&= -y[p_z, L_z] - [y, L_z]p_z + z[p_y, L_z] + [z, L_z]p_y \\
&= y[L_z, p_z] + [L_z, y]p_z - z[L_z, p_y] - [L_z, z]p_y \\
&= y(0) + (-i\hbar x)p_z - z(-i\hbar p_x) - (0)p_y \\
&= -i\hbar xp_z + i\hbar zp_x \\
&= i\hbar(zp_x - xp_z) \\
&= i\hbar L_y
\end{aligned}$$

Part (c)

Calculate the commutator of L_z and r^2.

$$\begin{aligned}
[L_z, r^2] &= -[r^2, L_z] \\
&= -[x^2 + y^2 + z^2, L_z] \\
&= -([x^2, L_z] + [y^2, L_z] + [z^2, L_z]) \\
&= -([xx, L_z] + [yy, L_z] + [zz, L_z]) \\
&= -\{(x[x, L_z] + [x, L_z]x) + (y[y, L_z] + [y, L_z]y) + (z[z, L_z] + [z, L_z]z)\} \\
&= -x[x, L_z] - [x, L_z]x - y[y, L_z] - [y, L_z]y - z[z, L_z] - [z, L_z]z \\
&= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y + z[L_z, z] + [L_z, z]z \\
&= x(i\hbar y) + (i\hbar y)x + y(-i\hbar x) + (-i\hbar x)y + z(0) + (0)z \\
&= i\hbar(xy + yx - yx - xy) \\
&= i\hbar(0) \\
&= 0
\end{aligned}$$

Calculate the commutator of L_z and p^2.

$$\begin{aligned}
[L_z, p^2] &= -[p^2, L_z] \\
&= -[p_x^2 + p_y^2 + p_z^2, L_z] \\
&= -([p_x^2, L_z] + [p_y^2, L_z] + [p_z^2, L_z]) \\
&= -([p_xp_x, L_z] + [p_yp_y, L_z] + [p_zp_z, L_z]) \\
&= -\{(p_x[p_x, L_z] + [p_x, L_z]p_x) + (p_y[p_y, L_z] + [p_y, L_z]p_y) + (p_z[p_z, L_z] + [p_z, L_z]p_z)\} \\
&= -p_x[p_x, L_z] - [p_x, L_z]p_x - p_y[p_y, L_z] - [p_y, L_z]p_y - p_z[p_z, L_z] - [p_z, L_z]p_z \\
&= p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y] + [L_z, p_y]p_y + p_z[L_z, p_z] + [L_z, p_z]p_z \\
&= p_x(i\hbar p_y) + (i\hbar p_y)p_x + p_y(-i\hbar p_x) + (-i\hbar p_x)p_y + p_z(0) + (0)p_z \\
&= i\hbar(p_xp_y + p_yp_x - p_yp_x - p_xp_y) \\
&= i\hbar(0) \\
&= 0
\end{aligned}$$

In preparation for part (d), also compute the commutator of L_x and p^2.

$$\begin{aligned}
[L_x, p^2] &= -[p^2, L_x] \\
&= -[p_x^2 + p_y^2 + p_z^2, L_x] \\
&= -([p_x^2, L_x] + [p_y^2, L_x] + [p_z^2, L_x]) \\
&= -([p_x p_x, L_x] + [p_y p_y, L_x] + [p_z p_z, L_x]) \\
&= -\{(p_x[p_x, L_x] + [p_x, L_x]p_x) + (p_y[p_y, L_x] + [p_y, L_x]p_y) + (p_z[p_z, L_x] + [p_z, L_x]p_z)\} \\
&= -p_x[p_x, L_x] - [p_x, L_x]p_x - p_y[p_y, L_x] - [p_y, L_x]p_y - p_z[p_z, L_x] - [p_z, L_x]p_z \\
&= p_x[L_x, p_x] \\
&\qquad + [L_x, p_x]p_x \\
&\qquad\qquad + p_y[L_x, p_y] \\
&\qquad\qquad\qquad + [L_x, p_y]p_y \\
&\qquad\qquad\qquad\qquad + p_z[L_x, p_z] \\
&\qquad\qquad\qquad\qquad\qquad + [L_x, p_z]p_z \\
&= p_x[yp_z - zp_y, p_x] \\
&\qquad + [yp_z - zp_y, p_x]p_x \\
&\qquad\qquad + p_y[yp_z - zp_y, p_y] \\
&\qquad\qquad\qquad + [yp_z - zp_y, p_y]p_y \\
&\qquad\qquad\qquad\qquad + p_z[yp_z - zp_y, p_z] \\
&\qquad\qquad\qquad\qquad\qquad + [yp_z - zp_y, p_z]p_z \\
&= p_x([yp_z, p_x] - [zp_y, p_x]) \\
&\qquad + ([yp_z, p_x] - [zp_y, p_x])p_x \\
&\qquad\qquad + p_y([yp_z, p_y] - [zp_y, p_y]) \\
&\qquad\qquad\qquad + ([yp_z, p_y] - [zp_y, p_y])p_y \\
&\qquad\qquad\qquad\qquad + p_z([yp_z, p_z] - [zp_y, p_z]) \\
&\qquad\qquad\qquad\qquad\qquad + ([yp_z, p_z] - [zp_y, p_z])p_z
\end{aligned}$$

Use the fact that $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$ and apply the known commutation relations.

$$\begin{aligned}
[L_x, p^2] &= p_x\{(y[p_z, p_x] + [y, p_x]p_z) - (z[p_y, p_x] + [z, p_x]p_y)\} \\
&\quad + \{(y[p_z, p_x] + [y, p_x]p_z) - (z[p_y, p_x] + [z, p_x]p_y)\}p_x \\
&\quad + p_y\{(y[p_z, p_y] + [y, p_y]p_z) - (z[p_y, p_y] + [z, p_y]p_y)\} \\
&\quad + \{(y[p_z, p_y] + [y, p_y]p_z) - (z[p_y, p_y] + [z, p_y]p_y)\}p_y \\
&\quad + p_z\{(y[p_z, p_z] + [y, p_z]p_z) - (z[p_y, p_z] + [z, p_z]p_y)\} \\
&\quad + \{(y[p_z, p_z] + [y, p_z]p_z) - (z[p_y, p_z] + [z, p_z]p_y)\}p_z \\
&= p_x\{(y \cdot 0 + 0 \cdot p_z) - (z \cdot 0 + 0 \cdot p_y)\} \\
&\quad + \{(y \cdot 0 + 0 \cdot p_z) - (z \cdot 0 + 0 \cdot p_y)\}p_x \\
&\quad + p_y\{(y \cdot 0 + i\hbar \cdot p_z) - (z \cdot 0 + 0 \cdot p_y)\} \\
&\quad + \{(y \cdot 0 + i\hbar \cdot p_z) - (z \cdot 0 + 0 \cdot p_y)\}p_y \\
&\quad + p_z\{(y \cdot 0 + 0 \cdot p_z) - (z \cdot 0 + i\hbar \cdot p_y)\} \\
&\quad + \{(y \cdot 0 + 0 \cdot p_z) - (z \cdot 0 + i\hbar \cdot p_y)\}p_z \\
&= p_y(i\hbar p_z) + (i\hbar p_z)p_y - p_z(i\hbar p_y) - (i\hbar p_y)p_z \\
&= i\hbar(p_y p_z + p_z p_y - p_z p_y - p_y p_z) \\
&= i\hbar(0) \\
&= 0
\end{aligned}$$

In preparation for part (d), also compute the commutator of L_y and p^2.

$$
\begin{aligned}
[L_y, p^2] &= -[p^2, L_y] \\
&= -[p_x^2 + p_y^2 + p_z^2, L_y] \\
&= -([p_x^2, L_y] + [p_y^2, L_y] + [p_z^2, L_y]) \\
&= -([p_x p_x, L_y] + [p_y p_y, L_y] + [p_z p_z, L_y]) \\
&= -\{(p_x[p_x, L_y] + [p_x, L_y]p_x) + (p_y[p_y, L_y] + [p_y, L_y]p_y) + (p_z[p_z, L_y] + [p_z, L_y]p_z)\} \\
&= -p_x[p_x, L_y] - [p_x, L_y]p_x - p_y[p_y, L_y] - [p_y, L_y]p_y - p_z[p_z, L_y] - [p_z, L_y]p_z \\
&= p_x[L_y, p_x] \\
&\qquad + [L_y, p_x]p_x \\
&\qquad\qquad + p_y[L_y, p_y] \\
&\qquad\qquad\qquad + [L_y, p_y]p_y \\
&\qquad\qquad\qquad\qquad + p_z[L_y, p_z] \\
&\qquad\qquad\qquad\qquad\qquad + [L_y, p_z]p_z \\
&= p_x[zp_x - xp_z, p_x] \\
&\qquad + [zp_x - xp_z, p_x]p_x \\
&\qquad\qquad + p_y[zp_x - xp_z, p_y] \\
&\qquad\qquad\qquad + [zp_x - xp_z, p_y]p_y \\
&\qquad\qquad\qquad\qquad + p_z[zp_x - xp_z, p_z] \\
&\qquad\qquad\qquad\qquad\qquad + [zp_x - xp_z, p_z]p_z \\
&= p_x([zp_x, p_x] - [xp_z, p_x]) \\
&\qquad + ([zp_x, p_x] - [xp_z, p_x])p_x \\
&\qquad\qquad + p_y([zp_x, p_y] - [xp_z, p_y]) \\
&\qquad\qquad\qquad + ([zp_x, p_y] - [xp_z, p_y])p_y \\
&\qquad\qquad\qquad\qquad + p_z([zp_x, p_z] - [xp_z, p_z]) \\
&\qquad\qquad\qquad\qquad\qquad + ([zp_x, p_z] - [xp_z, p_z])p_z
\end{aligned}
$$

Use the fact that $\left[\hat{A}\hat{B},\hat{C}\right]=\hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B}$ and apply the known commutation relations.

$$\begin{aligned}
[L_y,p^2] &= p_x\{(z[p_x,p_x]+[z,p_x]p_x)-(x[p_z,p_x]+[x,p_x]p_z)\} \\
&\quad + \{(z[p_x,p_x]+[z,p_x]p_x)-(x[p_z,p_x]+[x,p_x]p_z)\}p_x \\
&\quad + p_y\{(z[p_x,p_y]+[z,p_y]p_x)-(x[p_z,p_y]+[x,p_y]p_z)\} \\
&\quad + \{(z[p_x,p_y]+[z,p_y]p_x)-(x[p_z,p_y]+[x,p_y]p_z)\}p_y \\
&\quad + p_z\{(z[p_x,p_z]+[z,p_z]p_x)-(x[p_z,p_z]+[x,p_z]p_z)\} \\
&\quad + \{(z[p_x,p_z]+[z,p_z]p_x)-(x[p_z,p_z]+[x,p_z]p_z)\}p_z \\
&= p_x\{(z\cdot 0+0\cdot p_x)-(x\cdot 0+i\hbar\cdot p_z)\} \\
&\quad + \{(z\cdot 0+0\cdot p_x)-(x\cdot 0+i\hbar\cdot p_z)\}p_x \\
&\quad + p_y\{(z\cdot 0+0\cdot p_x)-(x\cdot 0+0\cdot p_z)\} \\
&\quad + \{(z\cdot 0+0\cdot p_x)-(x\cdot 0+0\cdot p_z)\}p_y \\
&\quad + p_z\{(z\cdot 0+i\hbar\cdot p_x)-(x\cdot 0+0\cdot p_z)\} \\
&\quad + \{(z\cdot 0+i\hbar\cdot p_x)-(x\cdot 0+0\cdot p_z)\}p_z \\
&= p_x(-i\hbar p_z)+(-i\hbar p_z)p_x+p_z(i\hbar p_x)+(i\hbar p_x)p_z \\
&= i\hbar(-p_xp_z-p_zp_x+p_zp_x+p_xp_z) \\
&= i\hbar(0) \\
&= 0
\end{aligned}$$

Part (d)

Assume that $H = (p^2/2m) + V$, where V only depends on r: $V = V(r)$. Calculate the commutator of H and L_x by letting it act on a test function.

$$\begin{aligned}
[H, L_x]f &= \left[\frac{p^2}{2m} + V, L_x\right] f \\
&= \left(\left[\frac{p^2}{2m}, L_x\right] + [V, L_x]\right) f \\
&= \left(\frac{1}{2m}[p^2, L_x] + [V, L_x]\right) f \\
&= \left(-\frac{1}{2m}[L_x, p^2] + [V, L_x]\right) f \\
&= \left(-\frac{1}{2m} \cdot 0 + [V, L_x]\right) f \\
&= [V, L_x]f \\
&= (VL_x - L_xV)f \\
&= VL_xf - L_x(Vf) \\
&= V(yp_z - zp_y)f - (yp_z - zp_y)(Vf) \\
&= V\left[y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right)\right] f - \left[y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right)\right] Vf \\
&= -i\hbar yV\frac{\partial f}{\partial z} + i\hbar zV\frac{\partial f}{\partial y} + i\hbar y\frac{\partial}{\partial z}(Vf) - i\hbar z\frac{\partial}{\partial y}(Vf) \\
&= \cancel{-i\hbar yV\frac{\partial f}{\partial z}} + \cancel{i\hbar zV\frac{\partial f}{\partial y}} + i\hbar y\frac{\partial V}{\partial z}f + \cancel{i\hbar yV\frac{\partial f}{\partial z}} - i\hbar z\frac{\partial V}{\partial y}f - \cancel{i\hbar zV\frac{\partial f}{\partial y}} \\
&= i\hbar\left(y\frac{\partial V}{\partial z} - z\frac{\partial V}{\partial y}\right) f \\
&= i\hbar\left(y\frac{dV}{dr}\frac{\partial r}{\partial z} - z\frac{dV}{dr}\frac{\partial r}{\partial y}\right) f \\
&= i\hbar\frac{dV}{dr}\left(y\frac{\partial r}{\partial z} - z\frac{\partial r}{\partial y}\right) f \\
&= i\hbar\frac{dV}{dr}\left[y \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2z - z \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y\right] f \\
&= i\hbar\frac{dV}{dr}(0)f \\
&= 0
\end{aligned}$$

Calculate the commutator of H and L_y by letting it act on a test function.

$$\begin{aligned}
[H, L_y]f &= \left[\frac{p^2}{2m} + V, L_y\right] f \\
&= \left(\left[\frac{p^2}{2m}, L_y\right] + [V, L_y]\right) f \\
&= \left(\frac{1}{2m}[p^2, L_y] + [V, L_y]\right) f \\
&= \left(-\frac{1}{2m}[L_y, p^2] + [V, L_y]\right) f \\
&= \left(-\frac{1}{2m}\cdot 0 + [V, L_y]\right) f \\
&= [V, L_y]f \\
&= (VL_y - L_yV)f \\
&= VL_yf - L_y(Vf) \\
&= V(zp_x - xp_z)f - (zp_x - xp_z)(Vf) \\
&= V\left[z\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial z}\right)\right] f - \left[z\left(-i\hbar\frac{\partial}{\partial x}\right) - x\left(-i\hbar\frac{\partial}{\partial z}\right)\right] Vf \\
&= -i\hbar zV\frac{\partial f}{\partial x} + i\hbar xV\frac{\partial f}{\partial z} + i\hbar z\frac{\partial}{\partial x}(Vf) - i\hbar x\frac{\partial}{\partial z}(Vf) \\
&= \cancel{-i\hbar zV\frac{\partial f}{\partial x}} + \cancel{i\hbar xV\frac{\partial f}{\partial z}} + i\hbar z\frac{\partial V}{\partial x}f + \cancel{i\hbar zV\frac{\partial f}{\partial x}} - i\hbar x\frac{\partial V}{\partial z}f - \cancel{i\hbar xV\frac{\partial f}{\partial z}} \\
&= i\hbar\left(z\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial z}\right) f \\
&= i\hbar\left(z\frac{dV}{dr}\frac{\partial r}{\partial x} - x\frac{dV}{dr}\frac{\partial r}{\partial z}\right) f \\
&= i\hbar\frac{dV}{dr}\left(z\frac{\partial r}{\partial x} - x\frac{\partial r}{\partial z}\right) f \\
&= i\hbar\frac{dV}{dr}\left[z\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x - x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2z\right] f \\
&= i\hbar\frac{dV}{dr}(0)f \\
&= 0
\end{aligned}$$

Calculate the commutator of H and L_z by letting it act on a test function.

$$\begin{aligned}
[H, L_z]f &= \left[\frac{p^2}{2m} + V, L_z\right] f \\
&= \left(\left[\frac{p^2}{2m}, L_z\right] + [V, L_z]\right) f \\
&= \left(\frac{1}{2m}[p^2, L_z] + [V, L_z]\right) f \\
&= \left(-\frac{1}{2m}[L_z, p^2] + [V, L_z]\right) f \\
&= \left(-\frac{1}{2m}\cdot 0 + [V, L_z]\right) f \\
&= [V, L_z]f \\
&= (VL_z - L_zV)f \\
&= VL_zf - L_z(Vf) \\
&= V(xp_y - yp_x)f - (xp_y - yp_x)(Vf) \\
&= V\left[x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)\right] f - \left[x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)\right] Vf \\
&= -i\hbar xV\frac{\partial f}{\partial y} + i\hbar yV\frac{\partial f}{\partial x} + i\hbar x\frac{\partial}{\partial y}(Vf) - i\hbar y\frac{\partial}{\partial x}(Vf) \\
&= \cancel{-i\hbar xV\frac{\partial f}{\partial y}} + \cancel{i\hbar yV\frac{\partial f}{\partial x}} + i\hbar x\frac{\partial V}{\partial y}f + \cancel{i\hbar xV\frac{\partial f}{\partial y}} - i\hbar y\frac{\partial V}{\partial x}f - \cancel{i\hbar yV\frac{\partial f}{\partial x}} \\
&= i\hbar\left(x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x}\right) f \\
&= i\hbar\left(x\frac{dV}{dr}\frac{\partial r}{\partial y} - y\frac{dV}{dr}\frac{\partial r}{\partial x}\right) f \\
&= i\hbar\frac{dV}{dr}\left(x\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial x}\right) f \\
&= i\hbar\frac{dV}{dr}\left[x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2y - y\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x\right] f \\
&= i\hbar\frac{dV}{dr}(0)f \\
&= 0
\end{aligned}$$

Therefore, the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r.